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Some non-self-similar solutions to a nonlinear diffusion equation

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Abstract. We generate exact solutions to the nonlinear diffusion equation $u_t = \nabla \cdot (u^{-1/2}\nabla u)$ which are not similarity solutions. Some applications and generalizations are indicated.

1. Introduction

It has been noted by Oron and Rosenau [1] and by Hill and Hill [2] that the nonlinear diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^{-1/2}\frac{\partial u}{\partial x}\right) \tag{1.1}$$

possesses solutions of the form

$$u = (a_0(x) + a_1(x)t)^2.$$

These solutions are not in general classical (group invariant) similarity solutions. The purpose of this note is to extend these results and to put them into a broader context.

We start with the multidimensional version of (1.1), namely

$$\frac{\partial u}{\partial t} = \nabla \cdot (u^{-1/2}\nabla u). \tag{1.2}$$

We note that applications of (1.2) arise in models for plasma diffusion [3] and for liquid helium [4]. If we write

$$u = w^2$$

then we obtain from (1.2) an equation in which the nonlinearity is of quadratic type, namely

$$w\frac{\partial w}{\partial t} = \nabla^2 w. \tag{1.3}$$

This may be contrasted with the usual way in which the more general equation

$$\frac{\partial u}{\partial t} = \nabla \cdot (u^m \nabla u) \tag{1.4}$$

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is written in the quadratically nonlinear form

$$\frac{\partial v}{\partial t} = v\nabla^2 v + \frac{1}{m}|\nabla v|^2 \tag{1.5}$$

by the substitution

$$u = v^{1/m}.$$

Forms such as (1.5) have been exploited by Galaktionov and Posashkov [5, 6] in deriving exact solutions to such equations by an approach which is slightly different from that adopted here, but which also relies on quadratic nonlinearities.

The generalization of (1.3) which follows from writing $u = w^{1/(m+1)}$ in (1.4) reads

$$w^{-m/(m+1)} \frac{\partial w}{\partial t} = \nabla^2 w$$

and this is not in general of the required form. Alternatively, we may note that substituting $u = w^{-1/m}$ into (1.4) yields

$$w^2 \frac{\partial w}{\partial t} = w\nabla^2 w - \left(\frac{1}{m} + 2\right)|\nabla w|^2$$

which in general contains cubic nonlinearities. The case $m = -\frac{1}{2}$ is special in that (1.4) may then be written in more than one form in which the nonlinearities are quadratic; here we shall exploit the form (1.3).

We may seek a solution to (1.3) of the form

$$w = a_0(\boldsymbol{x}) + a_1(\boldsymbol{x})t \tag{1.6}$$

which satisfies (1.3) provided that a_0 and a_1 satisfy

$$\nabla^2 a_1 = a_1^2 \tag{1.7}$$

$$\nabla^2 a_0 = a_0 a_1. \tag{1.8}$$

(The fact that this system of equations for a_0 and a_1 is not overdetermined results from the quadratically nonlinear form of (1.3).) Defining φ by the substitution $a_0 = \varphi a_1$ transforms (1.8) to

$$\nabla \cdot (a_1^2 \nabla \varphi) = 0 \tag{1.9}$$

where we have made use of (1.7). We shall now consider various special cases.

2. Radially symmetric solutions

Considering the case in which (1.2) is specialized in N-dimensions to

$$\frac{\partial u}{\partial t} = \frac{1}{r^{N-1}} \frac{\partial}{\partial r}\left(r^{N-1} u^{-1/2} \frac{\partial u}{\partial r}\right) \tag{2.1}$$

we arrive at

$$\frac{1}{r^{N-1}} \frac{\mathrm{d}}{\mathrm{d}r}\left(r^{N-1} \frac{\mathrm{d}a_1}{\mathrm{d}r}\right) = a_1^2 \tag{2.2}$$

$$\frac{1}{r^{N-1}} \frac{\mathrm{d}}{\mathrm{d}r}\left(r^{N-1} \frac{\mathrm{d}a_0}{\mathrm{d}r}\right) = a_0 a_1. \tag{2.3}$$

Equation (1.9) then implies that

$$\frac{\mathrm{d}\varphi}{\mathrm{d}r}=\frac{\beta_0}{r^{N-1}a_1^2}$$

where β_0 is an arbitrary constant of integration.

The general solution to (2.2) cannot be obtained explicitly for most N, but the special exact solution

$$a_1=2(4-N)r^{-2}$$

is easily derived. For $N\neq 6$ this leads to the general solution to (2.3) in the form

$$a_0=Ar^{-2}+Br^{4-N}$$

where A and B are arbitrary constants, while for $N=6$ we obtain

$$a_0=Ar^{-2}+Br^{-2}\ln r.$$

We may without loss of generality set $A=0$ by a translation of t, and we then obtain

$$N\neq 6 \qquad u=(Br^{4-N}+2(4-N)r^{-2}t)^2$$

$$N=6 \qquad u=(Br^{-2}\ln r-4r^{-2}t)^2$$

which give similarity solutions of the form

$$N\neq 6 \qquad u=t^{2(4-N)/(6-N)}f(r/t^{1/(6-N)})$$

$$N=6 \qquad u=\mathrm{e}^{-16t/B}f(r/\mathrm{e}^{4t/B}).$$

In general, however, solutions to (2.2) and (2.3) need not correspond to group invariant solutions of (2.1).

3. One-dimensional solutions

We now consider the one-dimensional case (1.1) discusssed in [1, 2]. We then have

$$\frac{\mathrm{d}^2a_1}{\mathrm{d}x^2}=a_1^2 \tag{3.1}$$

$$\frac{\mathrm{d}^2a_0}{\mathrm{d}x^2}=a_0a_1. \tag{3.2}$$

The general solution to (3.1), (3.2) may be obtained in the form

$$\int_{\beta_1}^{a_1}\frac{\mathrm{d}a}{(a^3-\alpha_1^3)^{1/2}}=\pm\sqrt{\tfrac{2}{3}}x$$

$$a_0=\alpha_0a_1\pm\sqrt{\tfrac{3}{2}\beta_0}a_1\int_{\beta_1}^{a_1}\frac{\mathrm{d}a}{a^2(a^3-\alpha_1^3)^{1/2}}$$

where α_0, α_1, β_0 and β_1 are arbitrary constants of integration, with

$$\frac{\mathrm{d}\varphi}{\mathrm{d}x}=\frac{\beta_0}{a_1^2}.$$

Part of the particular importance of this case lies in a non-local transformation which maps solutions to (1.1) of this form into mass preserving solutions of the equation

$$\frac{\partial U}{\partial T}=\frac{\partial}{\partial X}\left(U^{-3/2}\frac{\partial U}{\partial X}\right). \tag{3.3}$$

Such solutions are of importance in many applications. We seek a solution to (3.3) subject to the conditions

$$\text{as } |X|\to\infty \qquad U\to 0$$

such that the total mass

$$\int_{-\infty}^{\infty} U\,\mathrm{d}X = M$$

is a fixed constant. Writing

$$u=1/U \qquad x=\int_{-\infty}^{X} U(X',T)\,\mathrm{d}X' \qquad t=T \tag{3.4}$$

it may then be shown (as in [7]) that we obtain (1.1) subject to the conditions

$$\begin{aligned} &\text{as } x\to 0^+ && u\to+\infty \\ &\text{as } x\to M^- && u\to+\infty. \end{aligned} \tag{3.5}$$

In order for our solution

$$w=a_0(x)+a_1(x)t$$

to satisfy (3.5) we need

$$\int_{\alpha_1}^{a_1}\frac{\mathrm{d}a}{(a^3-\alpha_1^3)^{1/2}}=\sqrt{\frac{2}{3}}\left|x-\frac{M}{2}\right| \tag{3.6}$$

where the constant

$$\alpha_1\equiv a_1\left(\frac{M}{2}\right)$$

is given by

$$\int_{\alpha_1}^{\infty}\frac{\mathrm{d}a}{(a^3-\alpha_1^3)^{1/2}}=\frac{M}{\sqrt{6}}. \tag{3.7}$$

If we take $a_0\equiv 0$ then the solution would transform under (3.4) to the usual instantaneous source similarity solution to (3.3). However, this solution may be generalized to give a non-self-similar solution by taking $a_0\neq 0$. Hence we have

$$\begin{aligned} x<\frac{M}{2} &\qquad a_0=a_1\left(\alpha_0-\sqrt{\tfrac{3}{2}}\beta_0\int_{\alpha_1}^{a_1}\frac{\mathrm{d}a}{a^2(a^3-\alpha_1^3)^{1/2}}\right) \\ x>\frac{M}{2} &\qquad a_0=a_1\left(\alpha_0+\sqrt{\tfrac{3}{2}}\beta_0\int_{\alpha_1}^{a_1}\frac{\mathrm{d}a}{a^2(a^3-\alpha_1^3)^{1/2}}\right). \end{aligned}$$

By translating t by α_0 we may without loss of generality take $\alpha_0 = 0$ to give

$$x < \frac{M}{2} \qquad u = \left\{a_1\left(t - \sqrt{\tfrac{3}{2}}\beta_0 \int_{\alpha_1}^{a_1} \frac{da}{a^2(a^3 - \alpha_1^3)^{1/2}}\right)\right\}^2$$

$$x = \frac{M}{2} - \sqrt{\tfrac{3}{2}} \int_{\alpha_1}^{a_1} \frac{da}{(a^3 - \alpha_1^3)^{1/2}}$$

$$x > \frac{M}{2} \qquad u = \left\{a_1\left(t + \sqrt{\tfrac{3}{2}}\beta_0 \int_{\alpha_1}^{a_1} \frac{da}{a^2(a^3 - \alpha_1^3)^{1/2}}\right)\right\}^2$$

$$x = \frac{M}{2} + \sqrt{\tfrac{3}{2}} \int_{\alpha_1}^{a_1} \frac{da}{(a^3 - \alpha_1^3)^{1/2}}.$$

For the derivation of this solution to be consistent we require that

$$w = u^{1/2}$$

be the positive square root; in other words we need $w > 0$ which requires that $t > t_c$, where

$$t_c \equiv \sqrt{\tfrac{3}{2}}|\beta_0| \int_{\alpha_1}^{\infty} \frac{da}{a^2(a^3 - \alpha_1^3)^{1/2}}.$$

Because (1.1) is invariant under translations of t this constraint causes no difficulty in the interpretation of the solution.

We note that

$$\text{at } x = \frac{M}{2} \qquad u = \alpha_1^2 t^2 \qquad \frac{\partial u}{\partial x} = 2\beta_0 t$$

from which it follows that the inverse of the transformation (3.4) may be written as

$$U = 1/u \qquad X = \int_{M/2}^{x} u(x', t)\, dx' + \frac{2\beta_0}{\alpha_1} t \qquad T = t.$$

This expression for X may be shown to lead to

$$X = \left(\frac{a_0}{a_1}\right)^2 \frac{da_1}{dx} + 2\beta \frac{a_0}{a_1^2} \mp \sqrt{6}\beta_0^2 \int_{\alpha_1}^{a} \frac{da}{a^3(a^3 - \alpha_1^3)^{1/2}} + 2\frac{da_0}{dx} t + \frac{da_1}{dx} t^2$$

where the + sign holds for $x < M/2$ and the − sign holds for $x > M/2$; hence X may easily be written as a function of a_1 and t.

The singularity in the behaviour at $t = t_c$ takes the following form (we assume that $\beta_0 > 0$):

$$\text{as } x \to 0^+ \qquad a_1 \sim 6x^{-2} \qquad u \sim \left(\frac{\beta_0}{30}\right)^2 x^6 \qquad \text{at } t = t_c$$

giving

$$\text{as } X \to X_0^+ \qquad U \sim \left(\frac{\beta_0}{30}\right)^{-2/7} (7(X - X_0))^{-6/7} \qquad \text{at } T = t_c$$

where

$$X_0 \equiv \frac{2\beta_0}{\alpha_1} t_c - \int_0^{M/2} u(x', t_c)\, dx'.$$

This follows because

$$\text{as } x \to 0^+ \qquad X \sim X_0 + \frac{1}{7}\left(\frac{\beta_0}{30}\right)^2 x^7 \qquad \text{at } t = t_c.$$

We also have

$$\text{at } T = t_c \qquad X < X_0 \qquad U = 0.$$

Again taking $\beta_0 > 0$ we may obtain

$$\text{as } X \to +\infty \qquad U \sim (3X^2/2(T+t_c))^{-2/3}$$

$$\text{as } X \to -\infty,\ T > t_c \qquad U \sim (3X^2/2(T-t_c))^{-2/3}.$$

Finally, as $T \to +\infty$ the behaviour is given, as expected, by the instantaneous source similarity solution

$$U \sim T^{-2}(\alpha_1^3 + 3X^2/2T^4)^{-2/3}.$$

4. Multidimensional solutions

These solutions must satisfy (1.7) and (1.8). The difficulty in constructing solutions explicitly lies with (1.7); equation (1.8) is linear in a_0. Here we note two simple families of solutions in two dimensions.

(i) Taking the one-dimensional solution

$$a_1 = 6/x^2$$

equation (1.8) becomes

$$\frac{\partial^2 a_0}{\partial x^2} + \frac{\partial^2 a_0}{\partial y^2} = \frac{6a_0}{x^2}$$

from which many types of solution are easily constructed.

(ii) Taking the cylindrically symmetric solution

$$a_1 = 4/r^2$$

gives

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial a_0}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 a_0}{\partial \theta^2} = \frac{4a_0}{r^2}.$$

Writing $x^* = \ln r$, $y^* = \theta$ gives the Helmholtz equation

$$\frac{\partial^2 a_0}{\partial x^{*2}} + \frac{\partial^2 a_0}{\partial y^{*2}} = 4a_0.$$

Hence genuinely multidimensional solutions to (1.2) may easily be constructed using simple solutions to (1.7).

5. Discussion

There is a wide range of possible extensions to this approach, and we shall simply give some examples.

First, the inhomogeneous equation

$$\rho(x)\frac{\partial u}{\partial t}=\nabla\cdot(K(x)u^{-1/2}\nabla u)$$

evidently also has solutions of the form

$$u=(a_0(x)+a_1(x)t)^2. \tag{5.1}$$

Indeed the method generalizes to any equation of the form

$$\frac{\partial u}{\partial t}=\mathscr{L}[u^{1/2},x]$$

where the spatial operator $\mathscr{L}$ acts linearly on $u^{1/2}$. A further example is thus provided by the reaction–diffusion equation

$$\frac{\partial u}{\partial t}=\nabla\cdot(u^{-1/2}\nabla u)+\alpha u^{1/2}+\beta$$

where α and β are constants.

The nonlinear wave equation

$$\frac{\partial^2 u}{\partial t^2}=\nabla\cdot(u^{-1/2}\nabla u)$$

has solutions of the more general form

$$u=(a_0(x)+a_1(x)t+a_2(x)t^2)^2.$$

Writing $u=w^2$ gives

$$\frac{\partial}{\partial t}\left(w\frac{\partial w}{\partial t}\right)=\nabla^2 w$$

an equation with quadratic nonlinearities, and we obtain the system

$$\nabla^2 a_2=6a_2^2 \qquad \nabla^2 a_1=6a_1a_2 \qquad \nabla^2 a_0=a_1^2+2a_0a_2.$$

Another nonlinear wave equation

$$\frac{\partial^2 u}{\partial t^2}=\nabla\cdot(u^{-2/3}\nabla u)$$

becomes on writing $u=w^3$ an equation with cubic nonlinearities:

$$\frac{\partial}{\partial t}\left(w^2\frac{\partial w}{\partial t}\right)=\nabla^2 w$$

and admits solutions of the form

$$u=(a_0(x)+a_1(x)t)^3$$

with

$$\nabla^2 a_1=2a_1^3 \qquad \nabla^2 a_0=2a_0a_1^2.$$

In each case the form of solution is chosen to ensure that the governing system for the x-dependent coefficients is not overdetermined. It is hoped that such examples give some indication of the possible range of applications of techniques such as we have used here.

While our solutions are not in general group invariant solutions, they may be put in the framework of non-classical similarity solutions outlined by Olver and Rosenau [8]. Thus the solution (5.1) corresponds to a side condition

$$\frac{\partial^2}{\partial t^2}(u^{1/2})=0.$$

However, the difficulty with such approaches lies in recognizing which side conditions will lead to non-trivial solutions.

References

[1] Oron A and Rosenau P 1986 *Phys. Lett.* **118A** 172–6
[2] Hill D L and Hill J M 1990 *J. Eng. Math.* **24** 109–24
[3] Berryman J G and Holland C J 1978 *Phys. Rev. Lett.* **40** 1720–2
[4] Dresner L 1981 *Proc. 9th Symp. of Engineering Problems of Fusion Research* vol 1, ed C K Choi (Piscataway, NJ: IEEE) pp 618–21
[5] Galaktionov V A and Posashkov S A 1989 *USSR Comput. Math. Math. Phys.* **29** 112–9
[6] Galaktionov V A 1990 *Diff. Integral Eq.* **3** 863–74
[7] King J R 1989 *Q. J. Mech. Appl. Math.* **42** 537–52
[8] Olver P J and Rosenau P 1986 *Phys. Lett.* **114A** 107–12